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Brief Communication

Central extensions of Lie bialgebras and Poisson–Lie groups

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Abstract

We introduce the notions of central extensions of Lie bialgebras and Poisson–Lie groups, which we classify up to equivalence. Furthermore we describe the relation between these two notions in terms of our classification.

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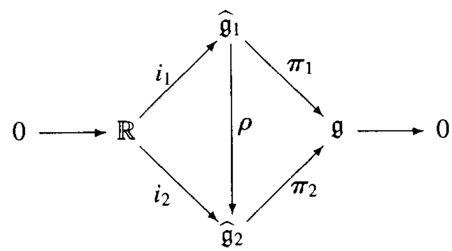
1. Central extensions of Lie bialgebras

In the following we denote by $\mathfrak{g}$ a finite dimensional real Lie bialgebra [1,2].

Definition 1.1. A Lie bialgebra $\widehat{\mathfrak{g}}$ is called a central extension of $\mathfrak{g}$ by $\mathbb{R}$ if there exists an exact sequence $0 \to \mathbb{R} \xrightarrow{i} \widehat{\mathfrak{g}} \xrightarrow{\pi} \mathfrak{g} \to 0$, where i and π are morphisms of Lie bialgebras such that $i(\mathbb{R})$ is a subset of the centre of the Lie algebra $\widehat{\mathfrak{g}}$. Two central extensions $\widehat{\mathfrak{g}}_1$ and $\widehat{\mathfrak{g}}_2$ of $\mathfrak{g}$ by $\mathbb{R}$ will be called equivalent if there exists an isomorphism of Lie bialgebras $\rho : \widehat{\mathfrak{g}}_1 \to \widehat{\mathfrak{g}}_2$ such that the following diagram commutes:



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In this section we describe explicitly the set $\mathrm{Ext_{big}}(\mathfrak{g},\mathbb{R})$ of all inequivalent Lie bialgebra central extensions of $\mathfrak{g}$ by $\mathbb{R}$. Another description of Lie bialgebra extensions (not necessarily central ones) in terms of derived functors can be found in Ref. [4]. We denote by $\mathcal{Z}^2(\mathfrak{g},\mathbb{R})$ the set of $\mathbb{R}$-valued 2-cocycles of the Lie algebra $\mathfrak{g}$ and by $\mathrm{Der}(\mathfrak{g}^*)$ the set of all derivations of the Lie algebra $\mathfrak{g}^*$. We will say that $\gamma \in \mathcal{Z}^2(\mathfrak{g},\mathbb{R})$ and $f \in \mathrm{Der}(\mathfrak{g}^*)$ are Drinfeld compatible if the transpose of f, denoted by $f^* : \mathfrak{g} \longrightarrow \mathfrak{g}$ [N.B. $(\mathfrak{g}^*)^* \cong \mathfrak{g}$ because $\mathfrak{g}$ is finite dimensional], satisfies the following: $\forall x,y \in \mathfrak{g}$: $f^*([x,y]) - [f^*(x),y] - [x,f^*(y)] = \mathrm{coad}_{\widetilde{\gamma}(y)}(x) - \mathrm{coad}_{\widetilde{\gamma}(x)}(y)$, where $\widetilde{\gamma} : \mathfrak{g} \to \mathfrak{g}^*$ is defined by $\langle \widetilde{\gamma}(x), y\rangle = \gamma(x,y)$, and where coad denotes the coadjoint action of the Lie algebra $\mathfrak{g}^*$ on its dual Lie algebra $\mathfrak{g}$.

Theorem 1.2. *There is a 1-1 correspondence between* $\mathrm{Ext_{big}}(\mathfrak{g},\mathbb{R})$ *and the quotient of* $\{(\gamma,f) \in Z^2(\mathfrak{g},\mathbb{R}) \times \mathrm{Der}(\mathfrak{g}^*) \mid \gamma, f$ *Drinfeld-compatible*$\}$ *by* $\{(\delta\phi, \mathrm{ad}_\phi) \mid \phi \in \mathfrak{g}^*\}$, *where* δ *denotes the coboundary operator in the* $\mathbb{R}$*-valued Lie algebra cohomology of* $\mathfrak{g}$, *and where* ad *is the adjoint action of the Lie algebra* $\mathfrak{g}^*$ *on itself.*

Proof (sketched). Since $\widehat{\mathfrak{g}}$ is in particular a Lie algebra central extension of $\mathfrak{g}$ by $\mathbb{R}$, we can identify $\widehat{\mathfrak{g}}$ with $\mathfrak{g}\times\mathbb{R}$, where the Lie bracket in $\mathfrak{g}\times\mathbb{R}$ is given by: $[(x,a),(y,b)] = ([x,y],\gamma(x,y))$, with $\gamma \in \mathcal{Z}^2(\mathfrak{g},\mathbb{R})$. One proves that the Lie bracket in $\widehat{\mathfrak{g}}^* \cong \mathfrak{g}^* \times \mathbb{R}$ is necessarily of the form: $[(\alpha,a),(\beta,b)] = ([\alpha,\beta] + af(\beta) - bf(\alpha),0)$ with $f \in \mathrm{Der}(\mathfrak{g}^*)$. The Drinfeld compatibility of γ and f is just the Drinfeld compatibility between the Lie brackets of the Lie bialgebra $\widehat{\mathfrak{g}}$, which justifies our terminology.

2. Central extensions of Poisson–Lie groups

In the sequel, let G be a Poisson–Lie group [1,3,5] and A a 1-dimensional abelian Poisson–Lie group.

Definition 2.1. A Poisson–Lie group $\widehat{G}$ is called a central extension of G by A if there exists an exact sequence $0 \longrightarrow A \xrightarrow{i} \widehat{G} \xrightarrow{\pi} G \longrightarrow 1$, where i and π are morphisms of Poisson–Lie groups such that $i(A)$ is contained in the centre of the group $\widehat{G}$. The notion of equivalent central extensions of Poisson–Lie groups is similar to that of Lie bialgebras.

We now want to describe $\mathrm{Ext_{L.P.}}(G,A)$, the set of all inequivalent central extensions of G by A. Let us denote by $\mathcal{Z}^2_e(G,A)$ the set of all A-valued 2-cocycles of the group G that are smooth around the identity element e of G. If $\phi \in \mathcal{Z}^2_e(G,A)$ is associated to the Lie group central extension $\widehat{G}$ of G by A (see [6]), then we can identify $\widehat{G}$ with $G\times A$ as groups where the product in $G\times A$ is given by: $(g,a)\cdot(h,b) = (gh, a+b-\phi(g,h))$. The adjoint action $\widehat{\mathrm{Ad}}$ of $\widehat{G}$ on its Lie algebra $\widehat{\mathfrak{g}} = \mathfrak{g}\times\mathbb{R}$ defines a map $M : G \longrightarrow \mathfrak{g}^*, g \mapsto M_g$ by:

$$\widehat{\mathrm{Ad}}_{(g,a)}(x,u) = (\mathrm{Ad}_g(x), u + M_g(x))\,; \quad (g,a) \in G\times A, (x,u) \in \mathfrak{g}\times\mathbb{R}.$$

One verifies that, if g is in a neigbourhood of e where ϕ is smooth, then $M_g = T_e(\widetilde{\phi}(g))$ (the tangent map to $\widetilde{\phi}(g)$ at e), where $\widetilde{\phi}(g) : G \to A$ is defined by: $(\widetilde{\phi}(g))(h) = \phi(g^{-1}, hg) - \phi(hg, g^{-1})$. One then proves that the 1-cocycles l and $\widehat{l}$ which define the Poisson–Lie structures of G and $\widehat{G}$ (see [1]), are related by:

$$\langle \widehat{l}(g,a), (\alpha, b) \otimes (\beta, c) \rangle = \langle l(g), \alpha \otimes \beta \rangle + \langle F(g), b\beta - c\alpha \rangle ;$$
$$(\alpha, b), (\beta, c) \in \mathfrak{g}^* \times \mathbb{R},$$

with $F \in C_e^\infty(G, \mathfrak{g})$, meaning that $F : G \to \mathfrak{g}$ is smooth around e. The fact that $\widehat{l}$ is a 1-cocycle implies the following relation, which we will call Poisson–Lie compatibility of ϕ and F:

$$\forall \alpha \in \mathfrak{g}^*, \forall g, h \in G : \langle F(gh), \alpha \rangle - \langle F(g), \alpha \rangle - \langle \mathrm{Ad}_g F(h), \alpha \rangle$$
$$= \langle \bigwedge^2 (\mathrm{Ad}_g) l(h), M_g \otimes \alpha \rangle .$$

The vanishing of the Schouten bracket of the Poisson bivector $\widehat{P}$ (defined by $\widehat{l}$) with itself is equivalent to the fact that $(T_e F)^* \in \mathrm{Der}(\mathfrak{g}^*)$.

Theorem 2.2. $\mathrm{Ext}_{\mathrm{L.P.}}(G, A)$ *is in bijection with the quotient of* $\{(\phi, F) \in \mathcal{Z}_e^2(G, A) \times C_e^\infty(G, \mathfrak{g}) \mid (T_e F)^* \in \mathrm{Der}(\mathfrak{g}^*)$, (ϕ, F) *Poisson–Lie compatible*$\}$ *by* $\{(\delta\chi, \Gamma_\chi) \mid \chi \in C_e^\infty(G, A)\}$, *where* δ *is the coboundary operator in the* A*-valued cohomology of* G, *and where* $\Gamma_\chi : G \to \mathfrak{g}$ *is defined by:* $\langle \Gamma_\chi(g), \alpha \rangle = l(g)((T_e\chi) \otimes \alpha), g \in G, \alpha \in \mathfrak{g}^*$.

3. The correspondence $\mathbf{Ext}_{\mathrm{L.P.}}(G, A) \to \mathbf{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$

If $0 \to A \xrightarrow{i} \widehat{G} \xrightarrow{\pi} G \to 1$ is a central extension of G by A, then $0 \to \mathbb{R} \xrightarrow{T_0 i} \widehat{\mathfrak{g}} \xrightarrow{T_{\hat{e}}\pi} \mathfrak{g} \to 0$ is a central extension of $\mathfrak{g}$ by $\mathbb{R}$. Hence we have a map $\mathrm{Ext}_{\mathrm{L.P.}}(G, A) \to \mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ which associates to each central extension of the Poisson–Lie group G by A the corresponding central extension of its Lie bialgebra $\mathfrak{g}$ by $\mathbb{R}$. Now, if $\phi \in \mathcal{Z}_e^2(G, A)$, then we can define $\gamma \in \mathcal{Z}^2(\mathfrak{g}, \mathbb{R})$ by:

$$\gamma(x, y) = \frac{d}{dt}\frac{d}{ds}\bigg|_{t=0, s=0} (\phi(\exp(tx), \exp(sy)) - \phi(\exp(sy), \exp(tx))) ,$$
$$\forall x, y \in \mathfrak{g}.$$

If we denote by (( )) the equivalence classes described in Thms. 1.2 and 2.2 then we have the following:

Proposition 3.1. *The correspondence* $\mathrm{Ext}_{\mathrm{L.P.}}(G, A) \to \mathrm{Ext}_{\mathrm{big}}(\mathfrak{g}, \mathbb{R})$ *is given by:*

$$((\phi, F)) \mapsto ((\gamma, (T_e F)^*)).$$

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